	Y13 Pure	CH11 11.10,11.11	Integration **Differential equations**	Lessons 3

Essential Knowledge Milestones

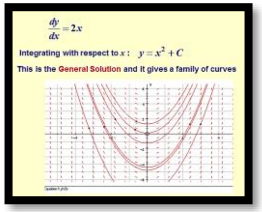
- be able to write a differential equation from a worded problem;
- be able to use a differential equation as a model to solve a problem;
- be able to solve a differential equation;
- be able to substitute the initial conditions or otherwise into the equation to find + c and the general solution.

Assumed Prior Knowledge/ Links / Interleaving

- A: Integration

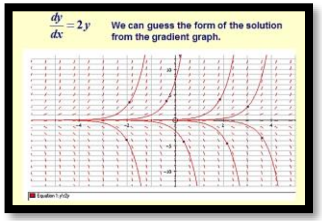
Teaching Points

Begin by considering the simplest possible differential equation (defined as first order) as below.



Notice that the graph drawing tool can plot the differential equation to give a family of curves which mirror the solution (family of parabolas)

The next differential equation is more difficult as we cannot integrate directly because the variable is *y* rather than x. But looking at the family of curves may give us a clue about the eventual solution.

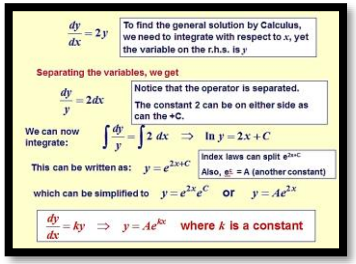


The curves look like exponentials.

The solution can be performed by using a method called 'separating variables', in which we rearrange and split up the $\frac{dy}{dx}$ as if it is a fraction. It is vital to keep all the *y*'s and dy's *and* the *x*'s and dx's together, but also the dx and dy must be in the numerator on each side.

Teaching Points

The full solution is shown below.



As suspected, the family of curves were exponential curves.

$y = Ae^{2x}$ is a general solution, but how do we find the value of the constant *A*? We need to have some information about the data from which the differential equation originates. Something along the lines of 'when x = 0, *y* = 2'.

Substituting this pair of values into the general solution and finding the value of *A*, will lead to a particular solution.

Sometimes we may have a choice of pairs to substitute or we may have two pairs of values in order to work out two constants.

Explain that questions may be set in a context and, in these cases, students need to interpret the solution of the differential equation in the context of the problem. This may including identifying limitations of the solution.

The following example is typical:-

> The population of a town was 50 000 in 2010 and had increased to 55 000 by 2015. Assuming that the population is increasing at a rate proportional to its size at any time, estimate the population in 2020 giving your answer to the nearest hundred.

$\frac{db}{dt} = kn \quad \Rightarrow \quad n = Ae^{kt}$ as above, but now *n* is the number of people and *t* is the time in years.

The validity of the solution for large values should be considered, for example, if the question was modelling population growth; would it be realistic for the value to keep increasing forever?

Potential Barriers to Access/Misconceptions

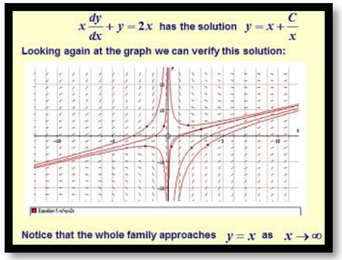
- Examiner comments indicate that this can prove a difficult topic for some students:
- When forming a differential equation some students wrote down the correct differential equation apparently fully understanding all the information given and interpreting it correctly. However, all sorts of errors abounded in other attempts, some not even involving a derivative, and some with derivatives in *x* and *y*.
- Many had a spurious *t* and/or *h*, either as a multiple or power, and the *k* appeared in a variety of places. Some students did not even form an equation, leaving a proportionality sign in their answer.
- When solving a differential equation most students knew they were expected to separate the variables and did it correctly, although there were some notation errors in the positioning of dx, at the front rather than the rear of the integrand. Those who failed to separate the variables, just produced nonsense. Many students struggled with the fact that integration by parts or substitution was needed. All students, no matter what their attempt at the integral, could obtain a method mark if they included a constant and tried to find it using the given initial conditions.

Questions & Prompts

- Find the general solution to $\frac{dy}{dx} = 6xy$
- Find the general solution to $\frac{dy}{dx} = \frac{1}{x+2}$
- The velocity, $v\ cms^{-1}$, of a particle is given by the differential equation $\frac{ds}{dt} = t^2 + t + 6$. find the initial displacement, $s\ cm$, of the particle, given that the displacement at 12s is 800cm.

Opportunities for Reasoning/Problem Solving/Proofs

The example below has a family of curves which has elements of both *y* = *x* and *y* = $\frac{1}{x}$, but it seems that the *y* = *x* is trying to win!



Also, for separating variables and finding the particular solution, encourage the more able students to use the initial conditions as the limits of integration, thus avoiding the + c.

Key Mathematical Vocabulary	
	Derivative, tangent, normal, turning point, stationary point, maximum, minimum, inflexion, parametric, implicit, differential equation, rate of change, product, quotient, first derivative, second derivative, increasing function, decreasing function.

Personal Development	Notes	Resources
Pupils are taught that they must show 'compassion' when working collaboratively as peer support may be required by those potentially making mistakes.		