THE ACADEMY OF ST NICHOLAS	**Y13 Pure**	**CH10** 10.1,10.2,10.3,10.4, 10.5,10.6	**Numerical Methods**	**Lessons 4**

Essential Knowledge Milestones

- be able to locate roots of f(x) = 0 by considering changes of sign of f(x);
- be able to use numerical methods to find solutions of equations.
- understand the principle of iteration;
- appreciate the need for convergence in iteration;
- be able to use iteration to find terms in a sequence;
- be able to sketch cobweb and staircase diagrams;
- be able to use cobweb and staircase diagrams to demonstrate convergence or divergence for equations of the form x = g(x).
- be able to solve equations approximately using the Newton-Raphson method;
- understand how the Newton-Raphson method works in geometrical terms.
- be able to use numerical methods to solve problems in context.

Assumed Prior Knowledge/ Links / Interleaving

- GCSE: Iterations and approximate areas under curves
- GCSE: Kinematics (velocity–time graphs)
- AS: Graphs, roots and functions
- AS: Differentiation and integration
- AS: Kinematics (velocity–time graphs)
- Sequences: understanding of limit, convergence, divergence
- Differentiation: differentiation of a range of functions required for the Newton-Raphson method
- Integration: the concept of integration as finding the area under a curve
- Cubic polynomials: using polynomial division some cubic equations can be solved by first factorising into linear and quadratic terms, for other cubic polynomials numerical methods are useful.
- Further numerical integration: using a combination of trapezium and midpoint rules to determine bounds for the answer correct to a desired level of accuracy

Teaching Points

- Students should be able to recognise that a root exists when there is a change of sign of f(x). Students should recognise this and remember it. There is often an easy mark missed on the exam for this because it is phrased slightly differently.
- Students should know that sign change is appropriate for continuous functions in a small interval.
- When the interval is too large the sign may not change as there may be an even number of roots.
- If the function is not continuous, the sign may change but there may be an asymptote (not a root) so the method will fail.
- The method at A level is to consider the roots of the function *y* = f(x) as the intersection of the two functions *y* = *x* and *y* = f(x) (hence *x* = f(x)).
- Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation *x* = f(x) and show how the convergence can be understood in geometrical terms by drawing cobweb and staircase diagrams
- Newton-Raphson method should be graphically represented as an introduction. Graph drawing packages are an essential way to 'look' at the curve and the potential position of the roots depending on the first approximation of the root.
- Recurrence relations, iterations and Newton-Raphson methods can be used obtain approximate solution(s) to an equation set in a context. The important point to make is that the original equation is too difficult to solve algebraically (e.g. the roots are decimal and/or the functions will not factorise or contain terms which are non-polynomials).
- The choice of degree of accuracy is dependent upon the context of the problem, e.g. nearest minute or number of years

Potential Barriers to Access /Misconceptions

- Students must define f(x) before substituting *x*-values to find a root.
- Most students can successfully identify the root of equations. However there are still many students who then write "change of sign therefore a root" without clarification of where the root lies and hence loose a mark.
- Marks are sometimes lost unnecessarily if students do not give their answers to the specified number of significant figures or decimal places.
- Giving a numeric solution without stating the error bounds
- Inability to link $x_{n+1} = g(x_n)$ relationships to their graphical interpretation; e.g. to explain why a starting value didn't lead to finding the root to which it was closest.
- Marks will be lost due to using degrees (instead of radians) if functions involve trigonometric terms.
- Choosing an unsuitable interval will also prevent progress in these questions.
- NR: Marks are often lost for sign errors and other numerical slips.
- Students must show full working leading to the correct answer for full marks. Giving a correct answer either without working or following wrong working will result in zero marks.

Opportunities for Reasoning/Problem Solving/Proofs

- Look at continuous functions and then contrast this with say $y = \frac{1}{x}$ and $y = \tan x$, which will not have any roots in some intervals despite a change of sign. Use graph drawing packages to investigate similar behaviour in other functions.
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- Prove that for all suitable starting values, the iteration $x_{n+1} = \sqrt[3]{x_n + \frac{3}{8}}$ converges to $\frac{1+\sqrt{13}}{4}$.
- Demonstrate the derivation of the Newton-Raphson formula.
- Show that if an iteration has a limit a, then a must be a root of the corresponding equation.
- Which iterations converge or diverge?
- Are there any values which cannot be substituted into certain iterations?
- Why does this staircase diagram fail?
- Consider, for example, iteration vs Newton–Raphson.
- Try different methods to find the roots of the same function. Which is the most efficient method or leads to the more accurate approximation?
- Which approximation method (when a choice is possible) gives the most efficient solution?
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Questions & Prompts

- Investigate the sequence $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$. [Why does this generate the square root of a?]
- Use three different numerical methods to find a root of the equation $x^3 + x = 5$
- Explain why some iteration processes result in cobweb diagrams whilst others result in a staircase diagram.
- Explain connections between the Newton-Raphson method and tangents to the curve.

Key Mathematical Vocabulary	Roots, continuous, function, positive, negative, converge, diverge, interval, derivative, tangent, chord, iteration, Newton-Raphson, staircase, cobweb, trapezium rule.

Personal Development	Notes	Resources
Pupils to learn that mathematicians achieve their 'personal best' through working collaboratively with other specialist mathematicians in order to interrelate more than one discipline.		
